Federal State Autonomous Educational Institution of Higher Education "Moscow Institute of Physics and Technology (National Research University)"

APPROVED
Head of the Phystech School of Applied Mathematics and Informatics
A.M. Raygorodskiy

Work program of the course (training module)

course:	Random Graphs. Part 2/Случайные графы. Часть 2
major:	Applied Mathematics and Informatics
specialization:	Advanced Methods of Modern Combinatorics/Продвинутые методы современной комбинаторики
	Phystech School of Applied Mathematics and Informatics
	Chair of Discrete Mathematics
term:	2
qualification:	Master

Semester, form of interim assessment: 3 (fall) - Exam

Academic hours: 75 AH in total, including:
lectures: 45 AH.
seminars: 30 AH.
laboratory practical: 0 AH.

Independent work: 75 AH.

Exam preparation: 30 AH.

In total: 180 AH, credits in total: 4

Authors of the program:

D.A. Shabanov, doctor of physics and mathematical sciences, associate professor, associate professor
M.E. Zhukovskiy, candidate of physics and mathematical sciences, associate professor

The program was discussed at the Chair of Discrete Mathematics 05.03.2020

Annotation

The course is devoted to the modern theory of random graphs. The questions of the presence of large structures in a random graph (large paths, matchings and cycles), the asymptotic behavior of the independence number and the chromatic number of a random graph, the laws of zero or one for random graphs are studied.

1. Study objective

Purpose of the course

mastering an advanced course in the theory of random counts

Tasks of the course

- mastering by students of basic knowledge (concepts, concepts, methods and models) in the field of random graphs;
- acquisition of theoretical knowledge and practical skills in the field of random graphs;
- providing advice and assistance to students in carrying out their own theoretical research in the field of random graphs.

2. List of the planned results of the course (training module), correlated with the planned results of the mastering the educational program

Mastering the discipline is aimed at the formation of the following competencies:

Code and the name of the competence	Competency indicators
UC-3 Able to organise and lead a team, developing a team strategy to achieve a goal	UC-3.2 Consider the interests, specific behavior, and diversity of opinions of team members/colleagues/counterparties
	UC-3.4 Plan teamwork, distribute tasks to team members, hold discussions of different ideas and opinions
Gen.Pro.C-1 Address current challenges in fundamental and applied mathematics	Gen.Pro.C-1.1 Apply fundamental scientific knowledge, new scientific principles, and research methods in applied mathematics and computer science
	Gen.Pro.C-1.2 Consolidate and critically assess professional experience and research findings
Gen.Pro.C-2 Iimprove upon and implement new mathematical methods in applied problem solving	Gen.Pro.C-2.1 Assess the current state of mathematical research within professional settings
Gen.Pro.C-4 Combine and adapt current information and communications technologies (ICTs) to meet professional challenges	Gen.Pro.C-4.2 Apply ICTs to solve the task in hand, to draw conclusions, and to evaluate the obtained results
Gen.Pro.C-5 An understanding of current scientific and technical problems in the field of informatics and computer technology, and is able to formulate professional tasks in scientific language	Gen.Pro.C-5.1 An understanding of the current state of research within his/her professional thematic area
	Gen.Pro.C-5.2 Able to assess the relevance of research in informatics and computer technology and its practical relevance
	Gen.Pro.C-5.3 A good command of the professional terminology used in modern scientific and technical literature, and is able to present the results of scientific work orally and in writing as part of professional communication
Gen.Pro.C-6 Capable of selecting and/or developing approaches to solving typical and new problems in informatics and computer technology, taking into account the characteristics and limitations of different solution methods	Gen.Pro.C-6.1 Able to analyse the problem, plan the solution, suggest and combine ways of solving it
	Gen.Pro.C-6.2 Capable of developing and upgrading software and hardware for information and automated systems
	Gen.Pro.C-6.3 Able to use research methods to solve new problems by applying knowledge from different fields of science (technology)
	Gen.Pro.C-6.4 Proficient in analytical and computational solution methods, and understands and takes into account in practice the limits of applicability of the solutions obtained

	Gen.Pro.C-6.5 Able to independently acquire, develop and apply mathematical, natural science, socio-economic and professional knowledge to solve non-standard problems, including in new or unfamiliar environments and in an interdisciplinary context

3. List of the planned results of the course (training module)

As a result of studying the course the student should:

know:

fundamental concepts, laws, theory of random graphs;
modern problems of the corresponding sections of random graphs;
concepts, axioms, methods of proofs and proofs of the main theorems in the sections included in the basic part of the cycle;
basic properties of the corresponding mathematical objects;
analytical and numerical approaches and methods for solving typical applied problems of random graphs.

be able to:

understand the task at hand;
use your knowledge to solve fundamental and applied problems of random graphs;
evaluate the correctness of the problem setting;
strictly prove or disprove the statement;
independently find algorithms for solving problems, including non-standard ones, and analyze them;
independently see the consequences of the results obtained;
Accurately represent mathematical knowledge in complex calculations, orally and in writing.

master:

skills of mastering a large amount of information and solving problems (including complex ones);
skills of independent work and mastering new disciplines;
culture of formulation, analysis and solution of mathematical and applied problems that require the use of mathematical approaches and methods of random graphs for their solution;
subject language of complex calculations and skills of competent description of problem solving and presentation of the results.

4. Content of the course (training module), structured by topics (sections), indicating the number of allocated academic hours and types of training sessions

4.1. The sections of the course (training module) and the complexity of the types of training sessions

№	Topic (section) of the course	Types of training sessions, including independent work			
		Lectures	Seminars	Laboratory practical	Independent work
1	Perfect matchings in a random graph	6	4		13
2	Random subgraphs of incomplete graphs	9	6		12
3	Distribution of degrees of vertices in a random graph	6	4		13
4	Paths and routes in graphs	9	6		12
5	Concentration inequalities in probability theory	6	4		13
6	Hamiltonian cycles in a random graph	9	6		12
AH in total		45	30		75
Exam preparation		30 AH.			
Total complexity		180 AH., credits in total 4			

4.2. Content of the course (training module), structured by topics (sections)

Semester: 3 (Fall)

1. Perfect matchings in a random graph

Distribution of degrees of vertices in a random graph

2. Random subgraphs of incomplete graphs

Janson's inequality, consequences from it. Azuma – Heffding inequality for martingales with bounded martingale differences

3. Distribution of degrees of vertices in a random graph

Method of moments. A sufficient condition for a random variable to be uniquely determined by its moments

4. Paths and routes in graphs

Threshold probabilities and threshold functions of possessing monotonic properties
a random subset.

5. Concentration inequalities in probability theory

Phase transition theorem in a random subgraph

6. Hamiltonian cycles in a random graph

A theorem on the existence of a threshold probability for an arbitrary monotone property of random subsets

5. Description of the material and technical facilities that are necessary for the implementation of the educational process of the course (training module)

Standard classroom.

6. List of the main and additional literature, that is necessary for the course (training module) mastering

Main literature

1. Случайные графы [Текст]/В. Ф. Колчин, -М., Физматлит, 2004
2. Модели случайных графов [Текст]/А. М. Райгородский, -М., МЦНМО, 2016
3. Модели случайных графов [Текст] : [учеб. пособие для вузов] / А. М. Райгородский ; Летняя школа "Современная математика", Дубна, июль 2008 г. — М. : МЦНМО, 2011 .— 136 с.

Additional literature

1. Графы. Алгоритмы на языке С [Текст] / В. В. Прут ; М-во образования и науки РФ, Моск. физ.-техн. ин-т (гос. ун-т) - М.МФТИ,2017

7. List of web resources that are necessary for the course (training module) mastering

http://dm.fizteh.ru/

8. List of information technologies used for implementation of the educational process, including a list of software and information reference systems (if necessary)

Multimedia technologies can be employed during lectures and practical lessons, including presentations.

9. Guidelines for students to master the course

It is recommended to successfully pass test papers, as this simplifies the final certification in the subject.

SUPPLEMENT

Assessment funds for course (training module)

major:	Applied Mathematics and Informatics
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Authors:

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1. Competencies formed during the process of studying the course

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2. Competency assessment indicators

As a result of studying the course the student should:

know:

fundamental concepts, laws, theory of random graphs;
modern problems of the corresponding sections of random graphs;
concepts, axioms, methods of proofs and proofs of the main theorems in the sections included in the basic part of the cycle;
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strictly prove or disprove the statement;
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independently see the consequences of the results obtained;
Accurately represent mathematical knowledge in complex calculations, orally and in writing.

master:

skills of mastering a large amount of information and solving problems (including complex ones);
skills of independent work and mastering new disciplines;
culture of formulation, analysis and solution of mathematical and applied problems that require the use of mathematical approaches and methods of random graphs for their solution;
subject language of complex calculations and skills of competent description of problem solving and presentation of the results.

3. List of typical control tasks used to evaluate knowledge and skills

Typical control tasks or other materials required to assess learning outcomes that characterize the stages of the formation of competencies.

Exam program

1. Perfect matchings in a random graph. The exact threshold probability of a perfect matching in the random graph G (n, p).
2. Paths and routes in graphs. Theorem on the length of the maximum path in a random graph G (n, p). The concept of a random two-color multigraph G (n, r, r), an algorithm for finding a path in a colored multigraph, its formal description.
3. Hamiltonian cycles in a random graph. Path transformations and Posch's lemma. Three lemmas on the existence of the property $| U \cup \Gamma (U) | \geq 3 | U |$ for small subsets U in the random graph G (n, p). A theorem on the limiting Hamiltonian property of a random graph G (n, p) under the condition np = (lnn + lnlnn + ω (n)) / n, where $\omega (n) \to +\infty$.
5. Concentration inequalities in probability theory. FKG inequality in the simplest case. Janson's inequality, consequences from it. Azuma-Heffding inequality for martingales with bounded martingale differences. Edge and vertex martingales in random graphs.
6. Independent sets in a random graph. The independence number α (G (n, p)) and its asymptotic behavior for p = const. Behavior of the independence number in the dynamic model of a random graph G (N, p).
7.Coloring of a random graph. An estimate of the probability of the absence of a large independence set in a random graph G (n, p) using Janson's inequality. Theorem on the asymptotic behavior of the chromatic number for the case p = const. Luchak's theorem on estimates for the chromatic number of a random graph G (n, p) in the general case (b / d).
8.Independent sets G (n, p) in the case p = c / n. Interpolation method and the law of large numbers for α (G (n, p)).
9.Independent sets G (n, p) in the case p = c / n. Karp-Sipser algorithm for finding an independent set in a graph, its application to trees. Approximation of a random graph by a random tree and finding the limiting constant for α (G (n, p)) for a fixed $c \leq 1$.
10. Chromatic number of a sparse random graph. The Ahlioptas-Naor theorem (b / d), the second moment method for estimating the threshold probability of r-colourability of a random graph.
11. First-order properties in random graphs. The laws of zero or one as min $[\![(p, 1-p)]\!]$ $n^{\alpha} \to +\infty$. Ehrenfeucht's theorem, a criterion for the validity of the law of zero or one.
12. Laws of zero or one in a random graph for $p = n^{(-\alpha)}$, $\alpha > 0$. Formulation of the Spencer-Shelah theorem, proof for all α, except for the rational ones from (0,1). Rigid and reliable pairs, rigid chains. The limitation of rigid chains.
13. Theorem on the number of reliable and maximum extensions (b / d). Ehrenfeucht's winning strategy used to prove the Spencer-Shelach theorem.

Examples of tasks for homework.

1) Let np = c> 1, c - fixed. Using Janson's inequality, prove that then with probability 1 - o (1 / ln n) the random graph contains a cycle of length $2\lceil \ln \ln n \rceil + 1$.
2) Prove that in a graph random process with probability tending to 1, the moments of disappearance of the last isolated vertex and the appearance of a perfect matching coincide.
3) Prove that the bipartition property of a random graph does not have an exact threshold probability.
4) Let $ y_n = 2np- \ ln n-2 \ ln \ ln n $, and $ X_n $ is the number of cherries in $ G (n, p) $. Prove that

\ begin {itemize}
\ item if $ y_n \ to- \ infty $ and $ n ^ 3p ^ 2 \ to + \ infty $, then $ X_n $ converges in probability to $ + \ infty $;
\ item if $ y_n \ to + \ infty $, then $ X_n $ converges in probability to zero.

\ end {itemize}
5) Prove that if $ p = n ^ {- \ alpha} $, $ k \ in \ mathbb {N} $, $ k \ geq 3 $, $ \ alpha \ in (0,1 / (k-1)) $, then the random graph $ G (n, p) $ obeys the $ k $ -law of zero or one (for any property $ L $ written using a first-order property whose quantifier depth does not exceed $ k $, the probability that the random graph $ G (n, p) $ has this property tends to either $ 0 $ or $ 1 $).
6) Let $ r \ geqslant 3 $ be a fixed natural number. Prove that for any constant $ c> 2r \ ln r - \ ln r $ the random graph $ G (n, c / n) $ is not $ r $ -colorable with high probability, that is, done
$$
{\ sf P} \ left (\ chi (G (n, c / n))> r \ right) \ to 1 \ mbox {for} n \ to \ infty.
$$

Evaluation formula
The student solves problems from homework, each problem gives 0.5 points to the assessment. The exam takes place in the form of an oral answer to 2 questions from the program of both parts of the course, directly on the exam you can score 4 points.

4. Evaluation criteria

1. Distribution of degrees of vertices in a random graph. Poisson limit theorem for the number of vertices of degree k in a random graph G (n, p). Similar theorems for the number of vertices of degree at least (at most) k. Theorems on the limiting concentration of the maximum and minimum degrees of vertices in a random graph G (n, p).
2. Connectivity of the random graph G (n, p). The theorem on the limiting probability of the connection G (n, p) under the condition p = (ln + c + o (1)) / n. A theorem on the exact threshold probability of the connection property G (n, p). Consequences of this theorem: the exact threshold probability for the property of the absence of isolated vertices, the threshold function for the connectedness of the random graph G (n, m).
3. Vertex and edge k-connectivity of graphs, separators in graphs. Lemma on separators in G (n, p). A theorem on the simultaneous occurrence of k-connectedness and the absence of vertices of degree less than k in a graph random process $\tilde{G}$.
4. Perfect matchings in a random graph. The exact threshold probability of a perfect matching in the random graph G (n, p).
5. Paths and routes in graphs. The Komlos – Szemerédi theorem on the length of the maximum path in a random graph G (n, p). The concept of a random two-color multigraph G (n, r, r), an algorithm for finding a path in a colored multigraph, its formal description.
6. Hamiltonian cycles in a random graph. Path transformations and Posch's lemma.
7. Hamiltonian cycles in a random graph. A theorem on the limit Hamiltonian property of a random graph G (n, p) under the condition p = (ln n + ln ln n + ω (n)) / n, where $\omega (n) \to + \infty$.

- the mark "excellent (10)" is given to a student who has shown comprehensive, systematized, in-depth knowledge of the curriculum of the discipline and the ability to confidently apply them in practice when solving specific problems, free and correct justification of the decisions

- the mark "excellent (9)" is given to a student who has shown comprehensive, systematized, in-depth knowledge of the curriculum of the discipline and the ability to apply them in practice in solving specific problems, free and correct justification of the decisions
- the mark "excellent (8)" is given to a student who has shown comprehensive systematized, deep knowledge of the curriculum of the discipline and the ability to apply them in practice in solving specific problems, and the correct justification of the decisions made
- the mark "good (7)" is given to a student if he firmly knows the material, expresses it competently and to the point, knows how to apply the acquired knowledge in practice, but makes some inaccuracies in the answer or in solving problems;
- the mark "good (6)" is given to a student if he knows the material, presents it competently and in essence, knows how to apply the knowledge gained in practice, but makes some inaccuracies in the answer or in solving problems;
- the mark "good (5)" is given to a student if he knows the material, and essentially expounds it, knows how to apply the knowledge gained in practice, but makes some inaccuracies in the answer or in solving problems;
- the mark "satisfactory (4)" is given to a student who has shown a fragmented, scattered nature of knowledge, insufficiently correct formulations of basic concepts, violation of the logical sequence in the presentation of the program material, but at the same time he owns the main sections of the curriculum necessary for further education and can apply the obtained knowledge by model in a standard situation;
- the mark "satisfactory (3)" is given to a student who has shown a fragmentary, scattered nature of knowledge, insufficiently correct formulations of basic concepts, violation of the logical sequence in the presentation of program material, but at the same time he has fragmentary knowledge of the main sections of the curriculum necessary for further education and can apply the knowledge gained by the model in a standard situation;
- the mark "unsatisfactory (2)" is given to a student who does not know most of the main content of the curriculum of the discipline, makes gross mistakes in the formulation of the basic concepts of the discipline and does not know how to use the knowledge gained in solving typical practical problems
- grade "unsatisfactory (1)" is given to a student who does not know the formulations of the basic concepts of the discipline.

5. Methodological materials defining the procedures for the assessment of knowledge, skills, abilities and/or experience

During the exam, students can use the discipline program.